Standing localized cluster in a continuum traffic model

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(Received 16 December 2000; published 12 April 2001)

We study the emergence of standing localized cluster in a continuum traffic model. The local-density profile, local velocity profile, and the phase boundary are obtained. The effect of the on ramp is discussed. The indication of the boundary induced phase transition is also discussed.

DOI: 10.1103/PhysRevE.63.056106 PACS number(s): 89.40.+k, 45.70.Vn, 47.54.+r, 05.40.−a

I. INTRODUCTION

The complex behavior of highway traffic near an on ramp has been attracting research interests recently [1–4]. Besides the well-known free flow and traffic jams, a new phase is identified and named synchronized flow [5–7]. The current theoretical research of the traffic flow is then focused on the characteristics of this new phase and the related hysterestic phase transitions [8–12]. Recent study proposed the possibility that instead of a single dynamical phase, this new phase may be a collection of multiple phases realized under different conditions [13,14]. The preliminary phase diagram is also constructed from model systems. Some of the phases have also been identified from the empirical data [15–17].

Among the various phases, the phase of standing localized cluster (SLC) is the most intriguing one and will be the focus of this paper. Far away from the ramp, a well-formed jam always propagates upstream, while near an on ramp, a localized cluster is observed to stay motionless. This distinguishing feature invites the name standing localized cluster, which is also known as pinned localized cluster. In the following paper, we study the emergence of SLC in a continuum model. In Sec. II, we study the density profile and the velocity profile. In Sec. III, we study the phase boundary. Discussions with a conclusion are in Sec. IV.

II. DENSITY PROFILE

In the continuum model of traffic flow, a hydrodynamic analogy is prescribed to the local density $\rho(x,t)$ and the local velocity $v(x,t)$. The system is governed by the continuity equation and the Navier-Stokes equation written in the following form

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0, \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=\frac{\rho}{\tau}[V(\rho)-v]-c_0^2\frac{\partial\rho}{\partial x}+\mu\frac{\partial^2 v}{\partial x^2}, \tag{2}$$

where $V(\rho)$ is the safe velocity and τ, c_0 and μ are constants related to the effects of relaxation, anticipation, and intrinsic dampening, respectively [18–20]. In this paper, the following parameters are adopted [13],

$$\tau=0.5 \text{ min},$$

$$c_0=54 \text{ km/h},$$

$$\mu=600 \text{ vehicles km/h},$$

$$V(\rho)=V_0\frac{1-\rho/\rho_0}{1+100(\rho/\rho_0)^4}, \tag{3}$$

where $V_0=120$ km/h and $\rho_0=140$ vehicles/km are the maximum values for velocity and density, respectively. In the stationary case, the safe velocity is achieved in a homogeneous flow and the flux $q(x,t)=\rho(x,t)v(x,t)$ becomes constant. With the above parameters, the maximum flux that can be achieved in the homogeneous flow is $q_{max}=2340$ vehicles/h. However, the homogeneous flow becomes unstable when the flux is larger than a critical value of $q_c=2250$ vehicles/h, which is revealed by the linear stability analysis [18]. Numerical works also find that the homogeneous flow is stable to large perturbations only when the flux is less than another critical value of $q_b=2060$ vehicles/h. When $q>q_b$, traffic jams begin to emerge and well-formed clusters moving with constant velocities are observed. Thus, for $q<q_b$, the homogeneous flow (free flow) is the only stable traffic state. For $q>q_c$, the moving cluster (traffic jam) becomes the only stable traffic state. For $q_b<q<q_c$, both the homogeneous flow and the moving cluster are stable. The transition between these two states can be induced by external perturbations. It is interesting to note that the analytical expressions for q_{max} and q_c are well known. However, to our knowledge, an analytical expression for q_b is still desired.

With the on ramp, a source term should be included in the continuity equation,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=q_{rmp}\phi(x), \tag{4}$$

where q_{rmp} denotes the incoming flux through the on ramp and $\phi(x)$ represents the spatial distribution of the flux. As the distribution $\phi(x)$ is localized at the ramp, a simple Gaussian is adopted,

$$\phi(x)=\frac{1}{\sqrt{2\pi}\;\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{5}$$

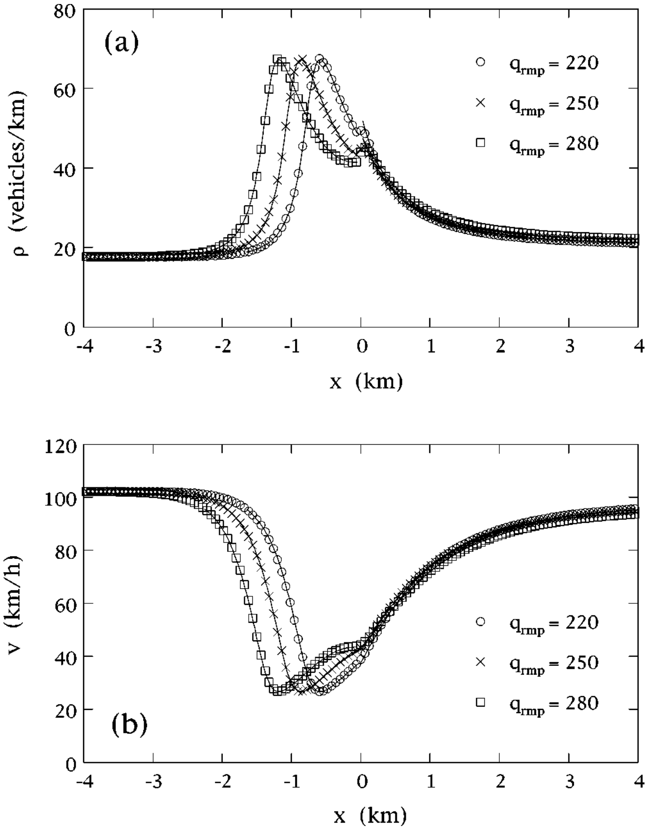

FIG. 1. (a) Density profile $\rho(x)$ at $q_{up}=1800$ vehicles/h for various values of q_{rmp}; $220<q_{rmp}<280$ vehicles/h. Solid lines are the results from Eq. (7). (b) The same as in (a) for the velocity profile $v(x)$.

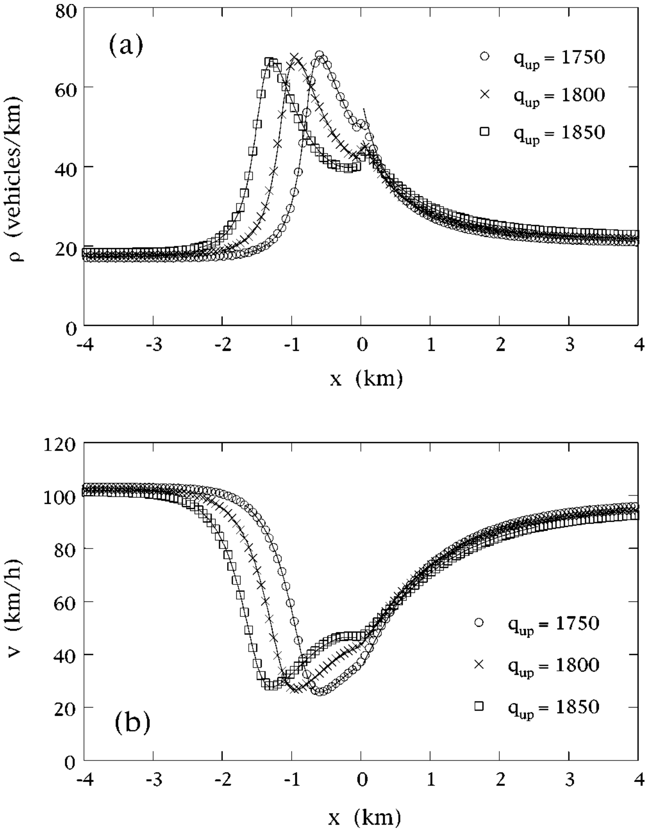

FIG. 2. (a) Density profile $\rho(x)$ at $q_{rmp}=260$ vehicles/h for various values of q_{up}; $1750<q_{up}<1850$ vehicles/h. Solid lines are the results from Eq. (7). (b) The same as in (a) for the velocity profile $v(x)$.

where σ denotes the length of the on ramp. In this paper, we perform extensive numerical simulations over a system of 60 km highway with an on ramp of $\sigma=60$ m right in the middle (position $x=0$). The controlled parameters are the on-ramp flux q_{rmp} and the upstream flux q_{up} at the boundary, i.e., 30 km upstream from the ramp. The system is prepared to start with a stationary state. Except for a small transition layer near the ramp, homogeneous flow is observed in both upstream and downstream. A finite perturbation is then introduced, which is a short pulse of traffic jam either through the ramp or from the downstream. A transition to a SLC state can then be triggered.

The SLC state can only be observed when q_{up} and q_{rmp} have a sum around the value of q_b. For a fixed value of q_{up} (q_{rmp}), the SLC state exists only within a small range of q_{rmp} (q_{up}). When the flux is too small, the cluster is unstable and disappears after a short time. When the flux is too large, the cluster is stable but not stationary. It moves to the upstream or oscillates around the ramp depending on the value of the flux. For example, at $q_{up}=1800$ vehicles/h, we have $220<q_{rmp}<280$ vehicles/h; at $q_{rmp}=260$ vehicles/h, we have $1750<q_{up}<1850$ vehicles/h. The typical behaviors of density and velocity profiles are shown in Figs. 1 and 2. The on-ramp flux introduces a discontinuity (at $x=0$) to the otherwise smooth profile. The discontinuity is much more abrupt presented in the density profile than in the velocity profile. A small bump located at $x=0$ is observed in the density profile. In contrast, only a change of the slope at $x=0$ is observed in the velocity profile. The profiles are independent of the parameter σ, except the small bump presented at $x=0$ of the density profile. As σ decreases, with fixed q_{up} and q_{rmp}, the bump becomes more abrupt.

With the discontinuity located exactly on the position of the on ramp, vehicles are clustered further upstream. The SLC state with vehicles jammed at the on ramp or downstream has never be observed. At a fixed upstream flux, the jammed cluster shifts toward upstream linearly with the increase of the on-ramp flux. Except for this obvious shift of the location, the detailed shape of the profile is independent of the on-ramp flux. After a shift along the x axis, the upstream profiles are scaled with the varying of the on-ramp flux. In contrast, in the case of fixed on-ramp flux, the scaling of the upstream profile can no longer be observed. The linear dependence of the profile shifting with the upstream flux increasing can still be observed. However, the amplitude decreases as the profile shifts toward upstream.

A detailed description of the stationary solution can be obtained by first imposing the constraints $\partial\rho/\partial t=\partial v/\partial t=0$. As the on ramp only constitutes a very small part of the highway, it is convenient to assume a localized limit of the spatial distribution $\phi(x)=\delta(x)$. The flux $q(x)=\rho(x)\ v(x)$ becomes constant but assumes different values for $x<0$ and $x>0$,

$$q=\begin{cases} q_{up} & \text{for} \quad x<0, \\ q_{up}+q_{rmp} & \text{for} \quad x>0. \end{cases} \tag{6}$$

The equation of motion in Eq. (2) can be rewritten as

$$\frac{d^2v}{dx^2}=\frac{q}{\mu}\left(1-\frac{c_0^2}{v^2}\right)\frac{dv}{dx}-\frac{q}{\mu\tau v}\left[V\left(\frac{q}{v}\right)-v\right]+\frac{c_0^2}{\mu v}q_{rmp}\delta'(x). \tag{7}$$

The problem can be reformulated as finding separate solutions in two semi-infinite regions, $x<0$ and $x>0$, with the appropriate matching conditions at $x=0$ [13],

$$v(0^+)=v(0^-),$$

$$\frac{d}{dx}v(0^+)=\frac{d}{dx}v(0^-)+\frac{c_0^2}{\mu v(0)}q_{rmp}. \tag{8}$$

The stationary local velocity $v(x)$ is prescribed to be continuous with a change of slope at $x=0$. The stationary local density then develops a discontinuity of $q_{rmp}/v(0)$ at $x=0$. The solution in each semi-infinite region can be obtained by analyzing the flow in the phase space (v,w) defined as

$$\frac{dv}{dx}=w,$$

$$\frac{dw}{dx}=\frac{q}{\mu}\left(1-\frac{c_0^2}{v^2}\right)w-\frac{q}{\mu\tau v}\left[V\left(\frac{q}{v}\right)-v\right]. \tag{9}$$

For $q<q_{max}$, there are two fixed points $(v,0)$ determined by

$$V\left(\frac{q}{v}\right)=v. \tag{10}$$

The fixed point with the larger velocity is a saddle point; the one with the smaller velocity is an attractor. The saddle point corresponds to the homogeneous flow in the initial conditions, which is unstable to perturbations. The desired solution can be obtained by finding a trajectory started from a saddle point with $q=q_{up}$, turned around by the attractor, and returned to a saddle point with $q=q_{up}+q_{rmp}$. With this trajectory, the velocity profile $v(x)$ can be easily converted. Then the density profile can be obtained by $\rho(x)=q/v(x)$. The numerical results can be well described, see Figs. 1 and 2. The scaling behavior presented in Fig. 1 becomes obvious as the flow in the phase space (v,w) follows the same trajectory at a fixed q_{up}. As q increases, the saddle point and the attractor move toward each other. The turnaround point shifts to a larger velocity. This results in the decreasing amplitudes observed in Fig. 2.

III. PHASE BOUNDARY

From the study of homogeneous highways without ramp, the phase boundary of SLC can be naively deduced. As a stable traffic jam always moves backward, the SLC state is often considered as a jam developed in downstream, moving through the ramp, and forbidden to exist in upstream, thus pinned at the ramp. The downstream flux must be large enough to support the traffic jams, while the upstream flux must be small enough to discourage the jams. Except near the ramp, the SLC state is basically a homogeneous flow; thus the downstream flux cannot be too large. The phase boundary is then obtained as follows:

$$q_b<q_{up}+q_{rmp}<q_c,$$

$$q_{up}<q_b. \tag{11}$$

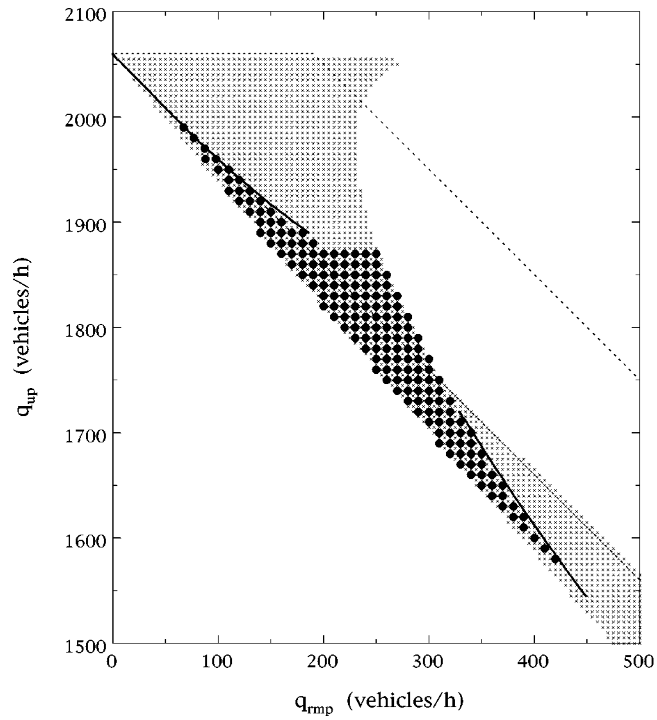

FIG. 3. Phase diagram of SLC in the (q_{up},q_{rmp}) plane. The data are the numerical results. Dashed lines are the naive expectation in Eq. (11). Shaded area are the results from Eq. (7). Solid lines are the results after imposing the constraints in Eq. (12).

However, the numerical results do not support such an expectation, see Fig. 3. Most of the SLC state has been observed outside of the expected region.

The approach of the last section also predicts a domain too large, see Fig. 3. However, the predicted region does include all the numerically observed SLC states. Thus the solutions from Eq. (7) can be taken as sufficient, though not necessary, conditions of SLC. Furthermore, as q_{up} and q_{rmp} increase, the onset of SLC can be correctly described. In the middle range of q_{up} or q_{rmp}, $1750<q_{up}<1870$ vehicles/h, or $250<q_{rmp}<310$ vehicles/h, the offset of SLC can also be described correctly. Within the above flux range, the numerical results can be fully reproduced by the solutions of Eq. (7). Outside this flux range, further constraints are required to reproduce the observed phase boundaries.

When $q_{rmp}<250$ or $q_{rmp}>310$ vehicles/h, SLC exists only for a very limited range of q_{up}. The solution of Eq. (7) can be obtained for quite a wide range of q_{up}, even beyond the limit set by the naive expectation in Eq. (11). The solution exists but is unstable, thus such a solution cannot be found numerically. Other stable states will become predominant. The correct boundaries of SLC can be reproduced by further introducing a constraint on the matching velocity $v(0)$ at the ramp. In the middle range of q_{rmp}, where the boundaries of SLC can be correctly predicted, the matching velocity $v(0)$ is limited to the following range:

$$32 < v(0) < 47 \;\; \text{km/h}. \tag{12}$$

When q_{rmp} is small, a solution with a much larger $v(0)$ can be found. The matching velocity is inversely proportional to the discontinuity of the density at the ramp. When $v(0)$ is too large, the developed discontinuity is not enough to pin down the cluster. The cluster becomes oscillating back and forth around the ramp, which is a different phase known as the recurring hump state. The correct boundary of SLC can be obtained by imposing a constraint on the upper limit of $v(0)$, see Fig. 3. It is interesting to observe that this upper limit is the same as in Eq. (12), i.e., $v(0) < 47$ km/h.

On the contrary, when q_{rmp} is large, a solution with a much smaller $v(0)$ can be found. When $v(0)$ is too small, the jump of the density over the ramp is too large and the SLC state becomes unstable. The cluster oscillates around the ramp with an increasing amplitude, which becomes another different phase known as the oscillating congested traffic state. Again, the correct boundary of SLC can be obtained by imposing a constraint on the lower limit of $v(0)$, see Fig. 3. We notice that this lower limit is the same as in Eq. (12), i.e., $v(0) > 32$ km/h. The correct phase boundaries can then be obtained by enforcing the condition in Eq. (12) over the solutions from Eq. (7).

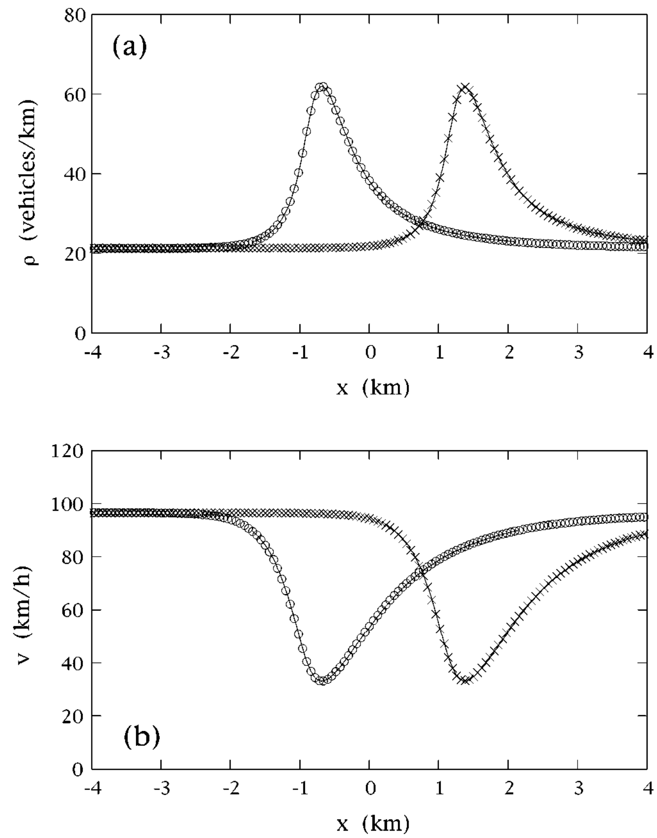

FIG. 4. (a) Density profile $\rho(x)$ at $q_{rmp}=5$ and $q_{up}=2055$ vehicles/h. The circle symbols are the SLC pinned at the ramp; the cross symbols are the SLC appeared in downstream, which have be shifted to a position near the ramp for comparison. Solid lines are the results from Eq. (7). (b) The same as in (a) for the velocity profile $v(x)$.

IV. DISCUSSIONS

In this paper, we study the standing localized cluster state within a continuum traffic model. We study the density profile, velocity profile, and the phase boundaries. The localized cluster is not standing right at the ramp, but slightly away to the upstream. Both the density and velocity profiles are mainly determined by the upstream flux. A scaling relation is observed. The discontinuity of local density locates right at the ramp, while only a change of slope can be observed in the local velocity. The numerical results can be well reproduced by the analysis using a delta function and appropriate matching conditions at the ramp. The phase boundaries in the middle range of the on-ramp flux can also be correctly reproduced. The phase boundaries in the whole flux range can be obtained by further imposing a constraint on the matching velocity, which is observed automatically in the middle range of the on-ramp flux.

The SLC state is often described as a characteristic of the on-ramp effect. Some conjectures even set a minimum to the required on-ramp flux. However, we would like to point out that the SLC state can also exist on a highway without ramp. With Eqs. (1) and (2), an interesting solution of SLC can be obtained at $q=q_b$. As it exists only at a unique value of the flux, this solution is often left unobserved in numerical study. In this solution, the location of the standing localized cluster is very sensitive to the details of the external perturbation. It can appear anywhere along the highway. In the system with an on ramp, the SLC state can be observed with very small on-ramp flux. The only requirement is that $q_{up} \sim (q_b - q_{rmp})$, see Fig. 3. The SLC states in systems without and with ramp are closely related. With a small on-ramp flux, the localized cluster can be easily pinned at the ramp. The typical results are shown in Fig. 4. With $q_{rmp}=5$ and $q_{up}=2055$ vehicles/h, a localized cluster is pinned at the ramp. As the on-ramp flux is small, the developed discontinuity in the density profile is not easy to discern. The effect of the on-ramp is revealed in the observation that the localized cluster has a fixed position. With a slightly different perturbation, the SLC has the same profile and the same position. In contrast, when a perturbation is introduced in the downstream, another SLC can be triggered to emerge at a position far away from the ramp. In this case, the downstream is basically a homogeneous system with $q=2060$ vehicles/h. With a slightly different perturbation, the SLC has the same profile but does not always appear in the same position. These two states have almost the same profile, see Fig. 4, and both can be described by the results in Eq. (7). With a ramp, a unique matching point is obtained; thus the position of the localized cluster is uniquely determined. Without a ramp, the matching point is not unique. In fact, the two trajectories are coincident and the whole trajectory can be the matching point. The position of the localized cluster is then left undetermined. With an on ramp, SLC can exist for a wider range of flux, but it can only appear near the ramp. Without a ramp, SLC exists only for a unique value of the flux, but it can appear anywhere. We note that SLC in the system without a ramp can also provide a precise definition of the critical flux q_b.

The naive expectation from the study of highway without a ramp sets a lower limit of flux to observe the SLC, i.e., $q_{up}+q_{rmp}>q_b$. Within this limit, the SLC state can only be observed in the middle range of flux, i.e., around $q_{up}=1850$ and $q_{rmp}=250$ vehicles/h. However, our paper reveals that most of the SLC state is observed with flux lower than such expectation. With $q_{up}+q_{rmp}<q_b$, the downstream flux is not large enough to support the formation of traffic jams. The free flow is the only stable solution in both upstream and downstream. The density and velocity profiles should be homogeneous except near the boundary. The success of the matching conditions at the ramp, in reproducing the profiles and phase boundaries, also indicates the SLC state can be taken as a boundary effect. In some cases, the trajectory of the downstream branch in the (v,w) plane is divergent beyond the matching point. Such a solution cannot be realized without a boundary. Though we place the on ramp right in the middle of the system configuration, the effect of the ramp is to provide a nontrivial boundary to the homogeneous solution. The transition to the SLC state can be taken as an example of the boundary induced phase transition.

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